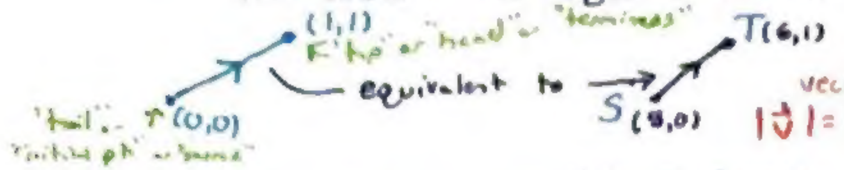
12.2: Vectors

Def: A vector in $\mathbb{R}^3$ is a directed line segment, where two vectors are equivalent when they are linear shifts



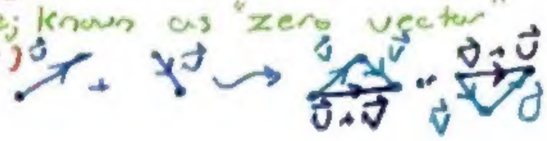
vectors parametric? $|\vec{v}| = \text{length}(\vec{ST}) = \sqrt{(S_x - T_x)^2 + (S_y - T_y)^2}$

vectors w/ same magnitude & direction are equivalent

Def: Magnitude, or length, of a vector $(\vec{v})$ is the length of an associated line

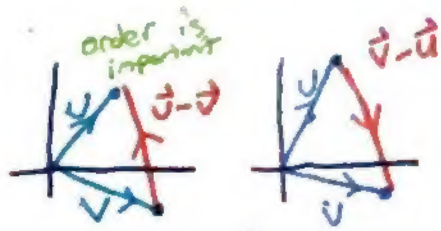
Operations on Vectors:

1. Take magnitude (vector $\mapsto$ real # ≥ 0)
   - At 0, head & tail the same; known as "zero vector"
2. Addition (vector + vector $\mapsto$ vector)
   - Tip - to - tail



"Parallelogram Law"

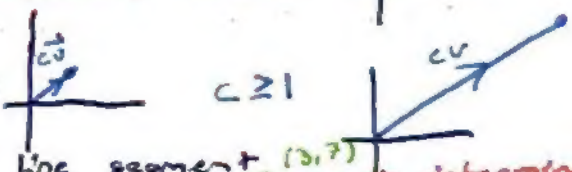
order is important



3. Subtraction (vector - vector $\mapsto$ vector)
4. Negate (vector $\to$ vector)
   - Flips the direction
5. Scalar multiplication (scalar $\cdot$ vector $\mapsto$ vector)
   - "Stretch the vector" (real #)

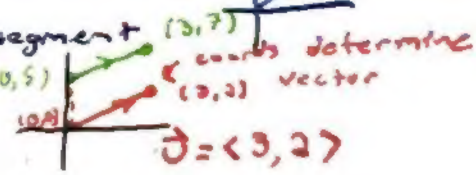
$c \geq 0$

$c < 0$: flips $\quad c < 1 \quad c \geq 1$



Components of Vectors

Nb: Every vector has a unique representative line segment w/ it's tail at the origin
- write in angle brackets $|\vec{v}| = \langle x, y, z \rangle$



Observation: A vector  has components "t - s"

Ex. The vector w/ directed line segment from (-2, 7) to (-3, 11) has components $\langle -3-(-2), 11-7 \rangle = \langle -1, 4 \rangle$

Ex. The Zero Vector has all components 0, denoted by $\vec{0} = \langle 0, 0 \rangle$ in $\mathbb{R}^2$

1. Magnitude $(\mathbb{R}^3)$ $\vec{v} \langle v_1, v_2, v_3 \rangle$, $|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$
2. Addition $\vec{u} = \langle u_1, u_2, u_3 \rangle + \vec{v} = \langle v_1, v_2, v_3 \rangle = \vec{u} + \vec{v} = \langle v_1 + u_1, v_2 + u_2, v_3 + u_3 \rangle$
3. Subtraction $\vec{u} = \langle u_1, u_2, u_3 \rangle - \vec{v} = \langle v_1, v_2, v_3 \rangle = u_1 - v_1 = \langle u_1 - v_1, u_2 - v_2, u_3 - v_3 \rangle$
4. Negation: $\vec{v} = \langle v_1, v_2, v_3 \rangle$ $\quad -\vec{v} = \langle -v_1, -v_2, -v_3 \rangle$
5. Scalar multiplication: $c \in \mathbb{R}$ $\vec{v} = (v_1, v_2, v_3)$ $\quad c\vec{v} = \langle cv_1, cv_2, cv_3 \rangle$

Properties of vector Operation:

Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ and $a, b \in \mathbb{R}$ all of the following hold

1. $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ (associativity)
2. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ (commutivity)
3. $\vec{0} + \vec{v} = \vec{v}$ (identity)
4. For all $\vec{u}$, there is a $-\vec{u}$ with a property that $\vec{u} + (-\vec{u}) = \vec{0}$ (negatives)
5. $a(b\vec{u}) = (ab)\vec{u}$ (regroup scalar)
6. $a\vec{v} + b\vec{v} = (a+b)\vec{v}$
7. $a\vec{u} + a\vec{v} = a(\vec{u} + \vec{v})$
8. $1\vec{u} = \vec{u}$; $0\vec{u} = \vec{0}$

Observation: $-\vec{u} = -1\vec{u}$

NB: ① scalar multiplicativity acts, a scalar
② Works in arbitrary dimensions

12.2 cont.

practice proof

Direction:

Properties of Magnitude (1 property):

Let $c \in \mathbb{R}$ and $\vec{v} \in \mathbb{R}^n$

$|c\vec{v}| = |c| \cdot |\vec{v}|$ ← magnitude of v

Observe: $|\vec{0}| = |0 \cdot \vec{v}| = |0||\vec{v}| = 0$

$|\vec{v}| = 0$ if and only if $\vec{v} = \vec{0}$

Def.: the direction of the vector $\vec{v}$ is the associated unit vector (ie vector w/ length 1).

direction of $\vec{v}$ is $\frac{1}{|\vec{v}|} \cdot \vec{v}$ when $\vec{v} \neq \vec{0}$

Claim: $\frac{1}{|\vec{v}|}\vec{v}$ is a unit vector

In Fact: $\left|\frac{1}{|\vec{v}|}\vec{v}\right| = \left|\frac{1}{|\vec{v}|}\right| |\vec{v}| = \frac{1}{|\vec{v}|}|\vec{v}| = 1$, when $v \neq 0$

Point: Components are important to algebra of vectors

In $\mathbb{R}^3$, there are 3 special vectors, called the component vectors:

$$\left.\begin{aligned} \vec{i} &= \langle 1,0,0 \rangle \\ \vec{j} &= \langle 0,1,0 \rangle \\ \vec{k} &= \langle 0,0,1 \rangle \end{aligned}\right\} \text{standard basis of } \mathbb{R}^3$$

Every vector is a sum of scalar multiples of components vectors

$$\begin{aligned} \vec{v} = \langle v_1, v_2, v^3 \rangle \\ &= \langle v_1, 0, 0 \rangle + \langle 0, v_2, 0 \rangle + \langle 0, 0, v_3 \rangle \\ &= v_1 \langle 1,0,0 \rangle + v_2 \langle 0,1,0 \rangle + v_3 \langle 0,0,1 \rangle \\ &= v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k} \end{aligned}$$